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Comments on "Asymptotic expansion of a Bessel function integral using hypergeometric functions" by L.J. Landau and N.J. Luswili

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Abstract

In a recent paper Landau and Luswili (*J. Comput. Appl. Math*. 132 (2001) 387) used generalized hypergeometric functions to obtain a complete asymptotic expansion for the integral $\int_0^{\pi/2} J_\mu(\lambda \sin\theta) J_\nu(\lambda \sin\theta)\, \mathrm{d}\theta$, where J_μ is the μth-order Bessel function of the first kind and λ is a large parameter tending to infinity. The purpose of this note is to point out that the same complete asymptotic expansion for this integral (as well as another one for a Hankel-type integral) has previously been obtained by Stoyanov et al. (*J. Comput. Appl. Math*. 50 (1994) 533) by using the same method. In addition, an alternative, simpler representation of the algebraic series contribution to the asymptotic expansion is provided. A few errors are also corrected and additional relevant references indicated.
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As pointed out in [12] and also discussed in [3], particular cases of the integral

$$S_{\mu\nu}(\lambda) = \int_0^{\pi/2} J_\mu(\lambda \sin\theta) J_\nu(\lambda \sin\theta)\, \mathrm{d}\theta \tag{1}$$

arise in various practical contexts, including wave scattering [10,11] and crystallography [9,14], and, more recently, in the analysis of optimally directive two-dimensional circular currents [8] and in the theory of a quantum particle on a lattice [3]. In these studies, asymptotic behavior of the representative integral for large values of parameter λ of certain physical content has been of particular interest. It is

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not surprising, therefore, that various techniques have been used over the years to obtain the requisite asymptotic expansions.

Briefly, an asymptotic expansion (comprising the leading and a few correction terms) for the particular case $\mu = \nu = 0$ was derived by Stoyanov and Farrell [10] by using a straightforward, heuristic approach. Their result was extended to $\mu = \nu > -1/2$ by Wong [14] (see also [9], [15, pp. 350–351]) by using a Mellin-transform technique. Then, Stoyanov et al. [11] used both the original straightforward approach and the Mellin-transform technique with certain simplifications and modifications to obtain the asymptotic expansions (to moderate orders) for integral $S_{\mu\nu}(\lambda)$ and for another, Hankel-type integral. Finally, at the suggestion of Temme [13] (see also [12, Acknowledgements] where Ref. [19] should be Ref. [21]), Stoyanov et al. [12] used the generalized hypergeometric and Meijer functions to obtain complete asymptotic expansions for the two integrals previously treated in [11].

Landau and Luswili [3], unaware of our later work [12], used generalized hypergeometric functions "to extend, simplify, and complete the analysis of Stoyanov and Farrell [10] and of Wong [14], as well as putting their considerations within a wider framework". Specifically, they begin their asymptotic analysis by expressing first the more general integral

$$S_{\mu\nu}^{ab}(\lambda) = \int_0^{\pi/2} \sin^a\theta \cos^b\theta \, J_\mu(\lambda\sin\theta) J_\nu(\lambda\sin\theta)\,\mathrm{d}\theta \tag{2}$$

in terms of the generalized hypergeometric function ${}_3F_4(-\lambda^2)$. It should be noted, however, that the representation [3, (22), (23)] that they have obtained for this integral can be deduced (as a particular case) from more general formulae tabulated in the literature (see e.g., [4, Eq. 13.3.2(19)] and [2, (10.45.1)]).

Although Landau and Luswili [3, ff. (23)] then stated that "the asymptotic expansion for large λ of $S_{\mu\nu}^{ab}(\lambda)$ is therefore obtained from the asymptotic expansion of ${}_3F_4(-\lambda^2)$", they do not actually work it out. Instead, as pointed out above, unaware of [12] they proceed with a special case $a = b = 0$, i.e., integral $S_{\mu\nu}(\lambda)$ defined in (1) to obtain its complete asymptotic expansion as the sum of the algebraic and exponential series according to [6, Section 5.11].

To obtain the algebraic asymptotic series of $S_{\mu\nu}(\lambda)$, Landau and Luswili [3, Section 2.2.1] start with [6, Eq. 5.11.1(7)] and then use a standard, well-documented [7] (see also [5], [6, Section 5.1]) procedure to deal with the case of two coinciding parameters in the representative hypergeometric function as a limit process. On the other hand, Stoyanov et al. [12, Section 3.3] employ the appropriate, ready formula of Luke [6, Eq. 5.1(29)], which has been deduced by L'Hôpital limit process, in order to obtain such an algebraic series. As a result, the contribution of the algebraic asymptotic series to $S_{\mu\nu}(\lambda)$ is expressed in somewhat formally different forms in [3, (31)] and [12, (3)]. Despite their different formal appearances, however, these two expressions for the algebraic asymptotic series of $S_{\mu\nu}(\lambda)$ are actually identical. This can be readily verified by utilizing well-known functional equations satisfied by the Gamma function, $\Gamma(z)$, and its logarithmic derivative, $\psi(z)$, [1, Sections 6.1, 6.3]. In particular, by using [1, Eqs. (6.1.15), (6.1.16)]

$$\Gamma(z+n) = \Gamma(z)\prod_{k=1}^{n}(z+k-1) \quad \text{and} \quad \Gamma(z-n) = \Gamma(z)\Bigg/\prod_{k=1}^{n}(z-k), \tag{3}$$

as well as [1, Eqs. (6.3.5), (6.3.6)]

$$\psi(z+n)=\psi(z)+\sum_{k=1}^{n}\frac{1}{z+k-1} \quad \text{and} \quad \psi(z-n)=\psi(z)-\sum_{k=1}^{n}\frac{1}{z-k}, \tag{4}$$

both expressions [3, (31)] and [12, (3)] for the algebraic asymptotic series of $S_{\mu\nu}(\lambda)$ can be written as

$$\sum_{n=0}^{\infty}\frac{\lambda^{-(2n+1)}\cos[\frac{\pi}{2}(\mu-\nu)]}{(-1)^n\pi(n!)^2}\left\{\prod_{k=1}^{n}\frac{1}{16}[(\mu+\nu)^2-(2k-1)^2][(\mu-\nu)^2-(2k-1)^2]\right\}$$
$$\times\left\{\ln\lambda-\psi_{\mu\nu}+\sum_{k=1}^{n}\left(\frac{4(2k-1)[(\mu^2+\nu^2-1)-4k(k-1)]}{[(\mu+\nu)^2-(2k-1)^2][(\mu-\nu)^2-(2k-1)^2]}+\frac{1}{k}\right)\right\}, \tag{5}$$

where $\prod_{k=1}^{n}=1$ and $\sum_{k=1}^{n}=0$ when $n=0$, and (cf. [12, (4b)])

$$\psi_{\mu\nu}=\gamma+\psi\left(\frac{1+\mu+\nu}{2}\right)+\frac{1}{2}\psi\left(\frac{1+\mu-\nu}{2}\right)+\frac{1}{2}\psi\left(\frac{1-\mu+\nu}{2}\right), \tag{6}$$

where γ is Euler's constant [1, Eq. 6.13]. Although not as compact as the corresponding expression in [12, (3)] where the Pochhammer symbols and other notations used in [6] have been employed, the alternative representation (5) is intuitively more appealing allowing one to see explicitly the particular n's contributions to the algebraic asymptotic series of $S_{\mu\nu}(\lambda)$.

The exponential asymptotic series obtained in [3, (47)] for $S_{\mu\nu}(\lambda)$ can be immediately identified with the corresponding term in the complete asymptotic expansion obtained in [12, (3)]. This is not surprising since the same formulae of Luke [6] have been used in both papers. Specifically [12, (3)] (cf. [3, (47), (38)]),

$$\sum_{n=0}^{\infty}\frac{\lambda^{-(n+3/2)}}{2\sqrt{\pi}}N_n\sin\left[2\lambda-\frac{\pi}{2}\left(\mu+\nu+\frac{1}{2}\right)-\frac{\pi}{2}n\right] \tag{7}$$

with the coefficients N_n determined from the four-term recursion formula

$$16(k+1)N_{k+1}=4\left[5k(k+1)-2\left(\mu^2+\nu^2-\frac{9}{8}\right)\right]N_k-8\left[k^3-k\left(\mu^2+\nu^2-\frac{1}{4}\right)\right]N_{k-1}$$
$$+\left[(\mu+\nu)^2-\left(k-\frac{1}{2}\right)^2\right]\left[(\mu-\nu)^2-\left(k-\frac{1}{2}\right)^2\right]N_{k-2}, \tag{8}$$

where $N_0=1$, and $N_m=0$ if $m<0$. Note that there is a sign error in the corresponding formula [3, (46)] as indicated explicitly below.

Remark 1. As pointed out in [12, Section 3.3] and explicitly shown in [11], the exponential (or, rather, oscillatory in λ, see (7) above) asymptotic contributions to $S_{\mu\nu}(\lambda)$ are due to the critical point (stationary phase point) at the upper limit of integration in (1).

Remark 2. Landau and Luswili [3, ff. (24)] point out that integral (1) is well defined if $\mathrm{Re}(\mu+\nu)>-1$, which they have supposed. Stoyanov et al. [12, at the end of Section 1] point out that this restriction, which is a sufficient condition for the validity of the complete asymptotic expansion [12, (3)], is to avoid the divergence of the integral at the lower limit of integration because of the singular behavior of the Bessel functions when their argument is zero and the indices (μ, ν) are negative (cf. [1, Eq. (9.1.14)]). They then emphasize that [12, (3)] as displayed is in fact valid for *all integer* (μ, ν) and indicate how this can be demonstrated.

Finally, a few, apparently typographical errors that have been noticed in Landau and Luswili's paper [3] are corrected here.

First, the term $(-1/4\pi\lambda^3)(\mu^2+\nu^2-1)\sin[(\pi/2)(\nu-\mu+1)]$ is missing in [3, (33)]; cf. (5) for $n=1$.

Second, in the last square brackets in the four-term recursion relation [3,(46)], instead of $-(k-\frac{1}{2})^4$ should be $+(k-\frac{1}{2})^4$; cf. (8).

Other minor obvious misprints in [3] are: in the Abstract the publication year of Wong's paper should be 1988 instead of 1998; on the right-hand side of [3, (11)] the leading subscript of ${}_{\mathrm{p}}F_{\mathrm{q}}$ is misplaced, and likewise for ${}_2F_3$ in [3, (19)]; the last term in [3, (27)] should be $(\alpha_1 \leftrightarrow \alpha_2)$ instead of $(\alpha_1 \rightarrow \alpha_2)$.

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